Rigidity and a common framework for mutually unbiased bases and k-nets

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Abstract

Many mysterious connections have been observed between collections of **mutually unbiased bases** (MUBs) and combinatorial designs called **k-nets** (particularly affine planes). We introduce the notion of a **k-net over an algebra**, providing a common framework for both objects, and derive a certain rigidity property which is new for MUBs. We specialize this result to a class of algebraically constructed MUBs and find as a corollary that certain large systems of this type cannot be completed.

What are MUBs?

Orthonormal bases $\mathcal{E}, \mathcal{F}$ of $\mathbb{C}^d$ are called ***unbiased*** if

$$|\langle \mathbf{e}, \mathbf{f} \rangle|^2 = 1/d \qquad \forall\, \mathbf{e} \in \mathcal{E}, \mathbf{f} \in \mathcal{F}.$$

A collection of r mutually unbiased bases (***MUBs***) is said to be ***complete*** if $r = d+1$ (as it is easy to prove that $r \leq d+1$).

MUBs arise naturally in several quantum information protocols and are of independent mathematical interest.

The eigenbases of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{L}(\mathbb{C}^2)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \beta_1 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \beta_2 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \beta_3 := \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

form a complete collection of MUBs.

Conjecture: Complete sets of MUBs exist in $\mathbb{C}^d$ if and only if d is a prime power.

What are k-nets?

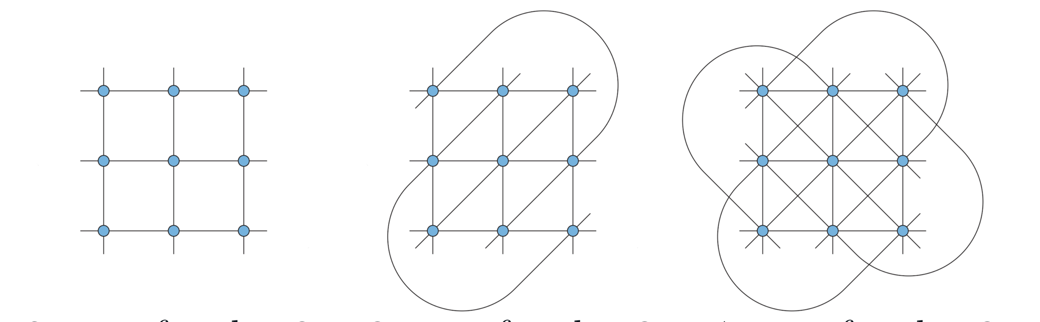
A ***k-net*** is an incidence structure consisting of a set X of *points* and a collection of subsets of X (called *lines*) such that

(i) the relation $||$ — where $\ell_1 || \ell_2$ means that $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$ — is an equivalence relation dividing the set of lines into k equivalence classes (called *parallel classes*);

(ii) any two lines are either parallel or intersect at a single point;

(iii) for any point p and line ℓ, $\exists$ a line parallel to ℓ containing p.

A *k-net of order d* consists of d^2 points and k parallel classes such that each class has exactly d lines and each line has exactly d points. A $(d+1)$-net of order d is called an *affine plane of order d*. Simple arguments show $k \leq d+1$ for all k-nets of order d, so one might say that affine planes are *complete* k-nets.



2-net of order 3   3-net of order 3   4-net of order 3 (complete 4-net) (affine plane of order 3)

Conjecture: Affine planes of order d exist if and only if d is a prime power.

C^*-algebra review

Recall that every C^*-algebra $\mathfrak{A}$ is $*$-isomorphic to the direct sum of full matrix algebras

$$\mathfrak{A} \cong \bigoplus_{j=1}^{k} M_{n_j}(\mathbb{C}),$$

for some $n_1, \ldots, n_k$ satisfying $\sum_{j=1}^k n_j^2 = \dim(\mathfrak{A})$, and all such $*$-isomorphisms are unitarily equivalent.

Generalized k-nets: a common framework

We now extend the notion of **classical k-nets** and **MUBs** to a more general setting.

Let $\mathfrak{A}$ be a finite-dimensional C^*-algebra with canonical normalized trace τ. We say that a collection of orthogonal projections $\mathcal{N} \subset \{P \in \mathfrak{A} \mid P^2 = P^* = P\}$ is a ***k-net over $\mathfrak{A}$*** if

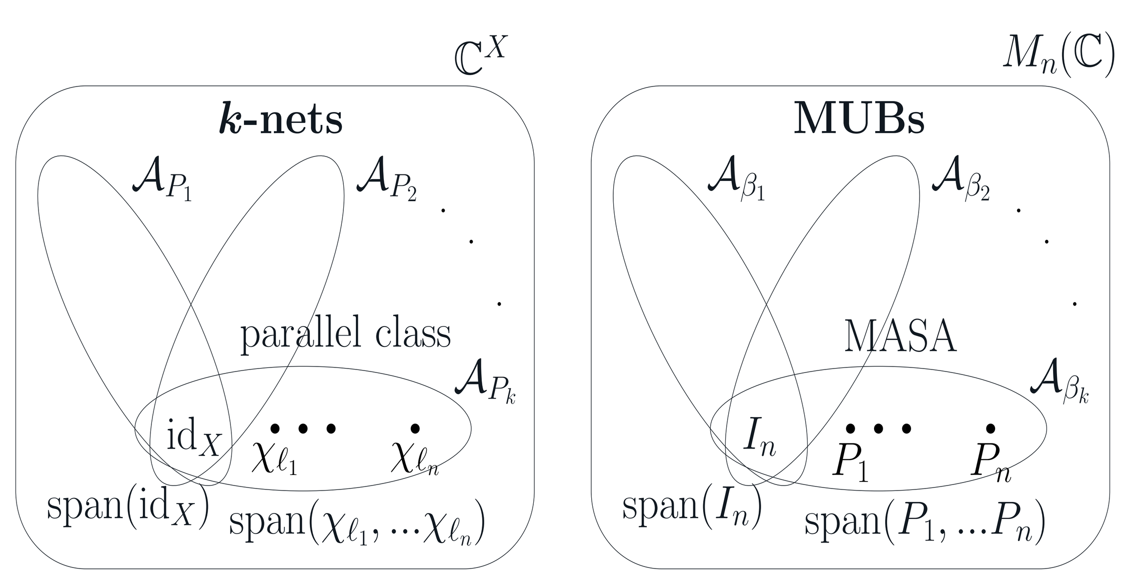
(i) the relation "$P = Q$ or $PQ = 0$" is an equivalence relation on $\mathcal{N}$ dividing $\mathcal{N}$ into k equivalence classes;

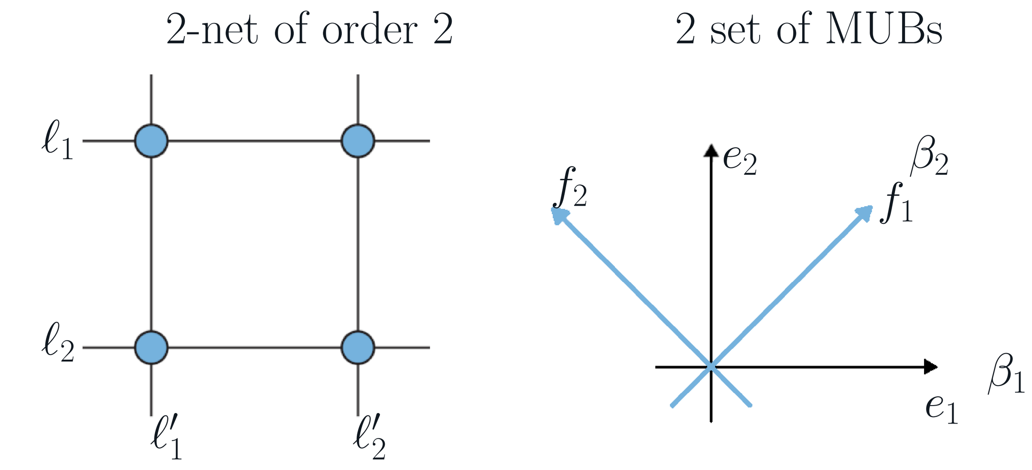
(ii) if $P, Q \in \mathcal{N}$ are not equivalent, then $\tau(PQ) = 1/\dim(\mathfrak{A})$;

(iii) the elements in each class sum to the identity I.

We refer to elements of $\mathcal{N}$ as ***lines*** and to the introduced equivalence classes as ***parallel classes***.

We can easily show (for $k \geq 3$) that each parallel class must have the same number d of lines (and $\tau(P) = 1/d$), so we say that $\mathcal{N}$ is a *k-net of order d*. This definition of k-nets over finite dimensional C^*-algebras generalizes the notions of both classical k-nets (case $\mathfrak{A} = \mathbb{C}^X$) and MUBs (case $\mathfrak{A} = M_d(\mathbb{C})$).





$\mathfrak{A} = \mathbb{C}^4$

$$\left.\begin{aligned} \chi_{\ell_1} &= (1,1,0,0) \\ \chi_{\ell_2} &= (0,0,1,1) \end{aligned}\right\} \mathcal{A}_{P_1} \qquad \left.\begin{aligned} \chi_{\ell'_1} &= (1,0,1,0) \\ \chi_{\ell'_2} &= (0,1,0,1) \end{aligned}\right\} \mathcal{A}_{P_2}$$

$\mathfrak{A} = M_2(\mathbb{C})$

$$\left.\begin{aligned} P_1 &= |e_1\rangle\langle e_1| \\ P_2 &= |e_2\rangle\langle e_2| \end{aligned}\right\} \mathcal{A}_{\beta_1} \qquad \left.\begin{aligned} Q_1 &= |f_1\rangle\langle f_1| \\ Q_2 &= |f_2\rangle\langle f_2| \end{aligned}\right\} \mathcal{A}_{\beta_2}$$

Rigidity Theorem

Our main result implies that generalized k-nets have a certain rigidity; they are determined by a proper subset of their parallel classes and cannot be "slightly modified" while retaining their defining properties.

Theorem 1. Let $\mathcal{N}$ be a k-net of order d over $\mathfrak{A}$, and suppose that $k \leq \sqrt{d}$. If $P = P^2 = P^* \in \mathrm{Span}(\mathcal{N})$ with $\tau(P) = \frac{1}{d}$, then $P \in \mathcal{N}$.

Corollary 1. Concretely, this means that sufficiently large combinatorial k-nets and sets of MUBs (of size $\geq d - \sqrt{d} + 1$) can be completed in at most one way.

References

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Nice MUBs

In quantum information theory, orthonormal bases of unitary matrices are fundamental to error correction and super-dense coding and are often constructed algebraically [2]:

Let G a group of order d^2 with identity e. A ***nice error basis*** with index group G is a set $\mathcal{E} = \{U(g) \mid g \in G\}$ of unitary operators in $M_d(\mathbb{C})$ such that

(i) $U(e) = I$,

(ii) $\mathrm{Tr}(U(g)) = 0$ for $e \neq g \in G$,

(iii) $U(g)U(h) = \lambda(g,h)U(gh)$ for all $g, h \in G$,

where $\lambda(g,h)$ is a complex phase factor.

Fix $d \geq 2$, and let $\omega = e^{2\pi i/d}$. We define X_d to be the cyclic shift matrix $X_d \mathbf{e}_j = \mathbf{e}_{j+1} \pmod d$ and Z_d to be the diagonal matrix $Z_d \mathbf{e}_j = \omega^{j-1}\mathbf{e}_j$, where $\{\mathbf{e}_j\}_j$ is the standard basis. Then, the *discrete Weyl operators* $\{X_d^j Z_d^\ell \mid (j,\ell) \in \mathbb{Z}_d^2\}$ form a nice error basis with index group $\mathbb{Z}_d^2$.

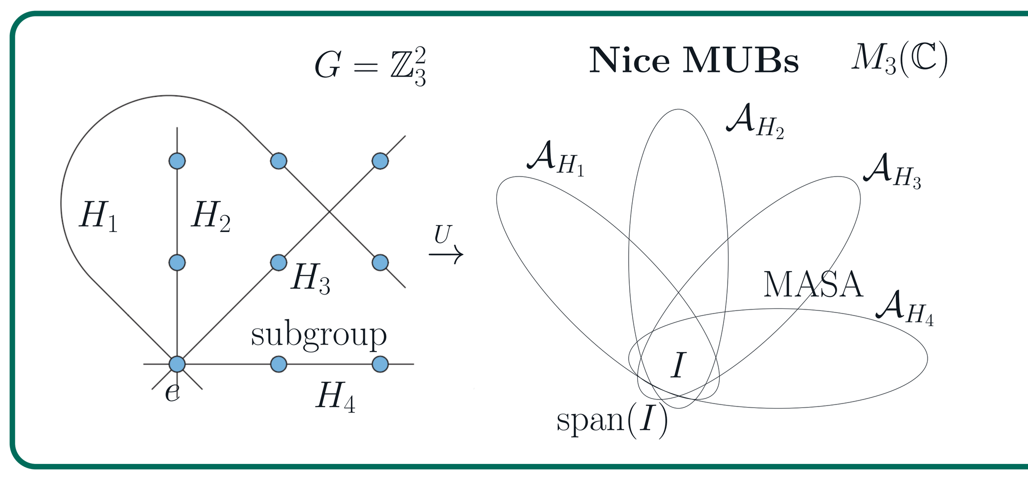
For each subgroup H of the index group G, define

$$\mathcal{A}_H := \mathrm{Span}\{U(h) \mid h \in H\}$$

to be the subspace of $M_d(\mathbb{C})$ spanned by the unitaries corresponding to H.

Proposition. Let $\mathcal{E}$ be a nice error basis for $M_d(\mathbb{C})$ with index group G, and take $H_1, \ldots, H_m$ to be subgroups of G of order d with pairwise trivial intersections. If, for each H_j, the associated unitaries of $\mathcal{E}$ are pairwise commuting, then $\mathcal{A}_{H_1}, \ldots, \mathcal{A}_{H_m}$ are quasi-orthogonal MASAs of $M_d(\mathbb{C})$, corresponding to a set of MUBs.

We call bases constructed from a nice error basis $\mathcal{E}$ in this way ***$\mathcal{E}$-nice mutually unbiased bases***, and can adapt Theorem 1 to this setting.



Theorem 2. Let $\mathcal{E}$ be a nice error basis for $M_d(\mathbb{C})$ with abelian index group. If an $\mathcal{E}$-nice set of at least $d+1-\sqrt{d}$ MUBs can be completed to a full set of $d+1$ MUBs, then this completion is unique and $\mathcal{E}$-nice.

Finally, we can use these rigidity results to prove that certain sets of MUBs cannot be completed.

Let $\mathcal{E}$ be a nice error basis for $M_d(\mathbb{C})$. A set of $\mathcal{E}$-nice MUBs in $\mathbb{C}^d$ is called ***weakly unextendible*** if there does not exist another mutually unbiased $\mathcal{E}$-nice basis.

Several examples of weakly unextendible MUBs are examined in [4, 3].

Corollary 2. A weakly unextendible set of at least $d+1-\sqrt{d}$ nice MUBs in $\mathbb{C}^d$ cannot be completed.

Concluding remarks

In [1], Bruck proved a uniqueness result which implies our rigidity theorem for classical k-nets. Moreover, he also proved an *existence result* stating that even larger k-nets automatically have a completion. Having examined uniqueness, we wonder whether one could derive such an existence result for MUBs using our framework.

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